Exam Quantum Field Theory
January 23, 2018
Start: 14:00h End: 17:00h

Each sheet with your name and student ID

INSTRUCTIONS: This is a closed-book and closed-notes exam. You are allowed to bring one A4 page written by you on one side, with useful formulas. The exam duration is 3 hours. There is a total of 9 points that you can collect.

NOTE: If you are not asked to **Show your work**, then an answer is sufficient. However, you might always earn more points by answering more extensively (but you can also lose points by adding wrong explanations). If you are asked to **Show your work**, then you should explain your reasoning and the mathematical steps of your derivation in full. Use the official exam paper for *all* your work and ask for more if you need.

USEFUL FORMULAS

The energy projectors for spin 1/2 Dirac fermions:

$$\sum_{r=1,2} u_r(\vec{p})\bar{u}_r(\vec{p}) = \frac{\not{p}+m}{2m}$$

$$\sum_{r=1,2} v_r(\vec{p})\bar{v}_r(\vec{p}) = \frac{\not{p}-m}{2m}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$\gamma^\mu\gamma_\mu = 4\mathbb{1} \qquad \gamma^\mu\not{p}\gamma_\mu = -2\not{p} \qquad \not{k}\not{p}\not{k} = 2(pk)\not{k} - k^2\not{p}$$

1. (2 points total) Given the lagrangian density for the complex scalar field

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - M^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$$

where the fields ϕ and its hermitian conjugate $\phi^\dagger$ are independent fields (analogously to two real scalar fields)

a) **[0.5 points]** Write the path integral Z in the presence of external sources.

b) **[1.5 points]** Derive the Feynman rule in momentum space for the interaction vertex with coupling λ. **Show your work**

Hints:

- The total lagrangian including external sources must be hermitian $\mathcal{L} = \mathcal{L}^\dagger$.
- The nonzero Wick contractions are $\underbracket{\phi^\dagger(x)\phi(y)} = \underbracket{\phi(x)\phi^\dagger(y)} = iD(x-y)$ with $iD(x-y)$ the usual scalar propagator, while $\underbracket{\phi(x)\phi(y)} = \underbracket{\phi^\dagger(x)\phi^\dagger(y)} = 0$.

2. (2 points total) The lagrangian density of Scalar QED describes the interactions of a complex scalar field with the electromagnetic field

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - M^2 \phi^\dagger \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \, ,$$

with the covariant derivatives

$$\begin{aligned} D_\mu \phi &= (\partial_\mu - ieA_\mu)\phi \\ D_\mu \phi^\dagger &= (\partial_\mu + ieA_\mu)\phi^\dagger \, . \end{aligned}$$

a) **[1 points]** Derive the equation of motion (EoM) for the fields ϕ and $\phi^\dagger$. **Show your work**

b) **[1 points]** According to Noether's theorem the invariance of the theory under the global $U(1)$ transformation

$$\phi'(x) = e^{i\alpha}\phi(x), \qquad \phi^{\dagger\prime}(x) = e^{-i\alpha}\phi^\dagger(x)$$

implies the existence of a conserved current J^μ, i.e. $\partial_\mu J^\mu = 0$. Find J^μ, whose general expression is given by Noether's theorem:

$$J^\mu = \sum_a \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_a} \delta \phi_a \tag{1}$$

where a runs over all independent fields.

Hint: Both the EoM and J^μ can be written in terms of the covariant derivatives.

3. (2 points total) Consider the Yukawa theory with lagrangian density

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \phi^2 + f \bar{\psi}\psi\phi$$

a) **[1 points]** For which spacetime dimensions d is the theory superrenormalizable? **Show your work**

b) **[1 points]** Provide an argument that shows that the amplitude with $E_B = 1$ and $E_F = 2$ is UV finite in two spacetime dimensions ($d = 2$).

Hints:

- The syperficial degree of divergence for the Yukawa theory can be written as $D = d - [g]V - [\phi]E_B - [\psi]E_F$ with $[\ldots]$ indicating the dimension of its argument, V the number of vertices and $E_{B,F}$ the number of external bosonic and fermionic lines, respectively.
- The amplitude in b) is the Yukawa interaction to all orders in perturbation theory.

4. (3 points total) Consider the two-body decay of a massive particle $1 \to 2$. Using the formula for the decay rate of a particle at rest with mass M into two particles with identical mass m

$$\Gamma = \frac{1}{4\pi}\frac{m^2}{M}\sqrt{1 - \frac{4m^2}{M^2}}\, X$$

with $X = \left(\mathcal{A}^\dagger \mathcal{A}\right)_{\text{unpol.}}$ the unpolarized squared amplitude,

a) **[2.5 points]** Evaluate Γ for the decay into an e^+e^- pair of a massive vector (spin 1) particle with interaction $g\bar{\psi}\gamma_\mu\psi A^\mu$. **Show your work**

b) **[0.5 points]** For a massive scalar (spin 0) particle with interaction $g\bar{\psi}\psi\phi$ (Yukawa theory) the analogous contribution X to the total decay rate for $\phi \to e^+e^-$ is

$$X = g^2\frac{M^2}{2m^2}\left(1 - \frac{4m^2}{M^2}\right) . \tag{2}$$

What is the maximum value of m/M for which $\Gamma_{spin\,0} \geq \Gamma_{spin\,1}$? **Show your work**

Hints:

- Use for the sum over the polarizations of the massive vector particle

$$\sum_{a=1}^{3} \epsilon_a^\mu(k)\epsilon_a^\nu(k) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{M^2}$$

with ϵ_μ real.

- You should find that X can be written in terms of the $e^-(e^+)$ mass m and M only.